

Homomorphisms

When we talk about functions between groups, it makes sense to limit our scope to functions that preserve the group operation (morphisms in the category of groups). More precisely:

Def: Let G and H be groups. A function $\varphi: G \rightarrow H$ is a homomorphism if $\forall x, y \in G, \varphi(xy) = \varphi(x)\varphi(y)$.

\uparrow multiplied in G \uparrow multiplied in H

Ex: Consider the function $\varphi: \mathbb{Z} \rightarrow D_{2n}$ defined w/ addition

$$\varphi(a) = r^a (= r^{a \pmod{n}}).$$

Then for $a, b \in \mathbb{Z}$, $\varphi(a+b) = r^{a+b} = r^a r^b = \varphi(a)\varphi(b)$, so it is a homomorphism.

Ex: Consider the function $\varphi: D_8 \rightarrow \mathbb{Z}/4\mathbb{Z}$ defined

$$\varphi(s^i r^j) = \bar{j}.$$

Is this a homomorphism?

$$\varphi(sr) + \varphi(sr) = \bar{1} + \bar{1} = \bar{2}, \text{ but}$$

$$\varphi(sr sr) = \varphi(s r r^3 s) = \varphi(1) = \bar{0}, \text{ so it's not a homomorphism.}$$

Ex: Define the map $\exp: \mathbb{R} \rightarrow \mathbb{R}^+$ by $\exp(x) = e^x$.
w/ addition \uparrow w/ multiplication
(check that this is a group!)

Then $\exp(x+y) = e^{x+y} = e^x e^y = \exp(x)\exp(y)$, so it's a homomorphism.

In fact, it's a bijection as well! We have an inverse homomorphism:

$\ln: \mathbb{R}^+ \rightarrow \mathbb{R}$, the natural logarithm.

Notice that this means \mathbb{R}^+ and \mathbb{R} have the same set of elements (renamed), and the same exact group structure. That is, they are "isomorphic":

Def: An isomorphism of groups is a bijective homomorphism.

If $\varphi: G \rightarrow H$ is an isomorphism, we say G and H are isomorphic, denoted $G \cong H$.

Note that if G is any group, the identity $\text{id}: G \rightarrow G$ is an isomorphism, but not necessarily the only isomorphism $G \rightarrow G$:

Ex: Define $\varphi: \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z}$ by

$$\begin{aligned} 0 &\mapsto 0 \\ 1 &\mapsto 3 \\ 2 &\mapsto 2 \\ 3 &\mapsto 1 \end{aligned}$$

This is a nonidentity isomorphism.

In fact:

Claim: G an abelian group, $\varphi: G \rightarrow G$ defined $\varphi(x) = x^{-1}$ is an isomorphism. (In fact, if and only if!)

Pf: $\varphi(xy) = (xy)^{-1} = x^{-1}y^{-1} = \varphi(x)\varphi(y)$. \square
see HW

Note: If $G \cong H$, then any property of G that depends only on the group structure will also hold for H . e.g.

- $|G| = |H|$
- G is abelian $\Leftrightarrow H$ abelian
- $x \in G, |x| = |\varphi(x)|$
- $G' \leq G \Leftrightarrow \varphi(G') \leq H$.

Ex: The quaternion group, Q_8 is defined

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

where • $1 \cdot a = a \cdot 1 = a \quad \forall a \in Q_8$

• $(-1)(-1) = 1, (-1) \cdot a = a \cdot (-1) = -a \quad \forall a \in Q_8$.

• $i^2 = j^2 = k^2 = -1$

• $i \cdot j = k, j \cdot k = i, k \cdot i = j$

• $j \cdot i = -k, k \cdot j = -i, i \cdot k = -j$

Note that $|i| = |j| = |k| = 4$. In particular, no element has order 8, so $Q_8 \not\cong \mathbb{Z}/8\mathbb{Z}$.